Hand Out - 3

In this handout, we begin by considering a linear transformation

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longmapsto Ax$$

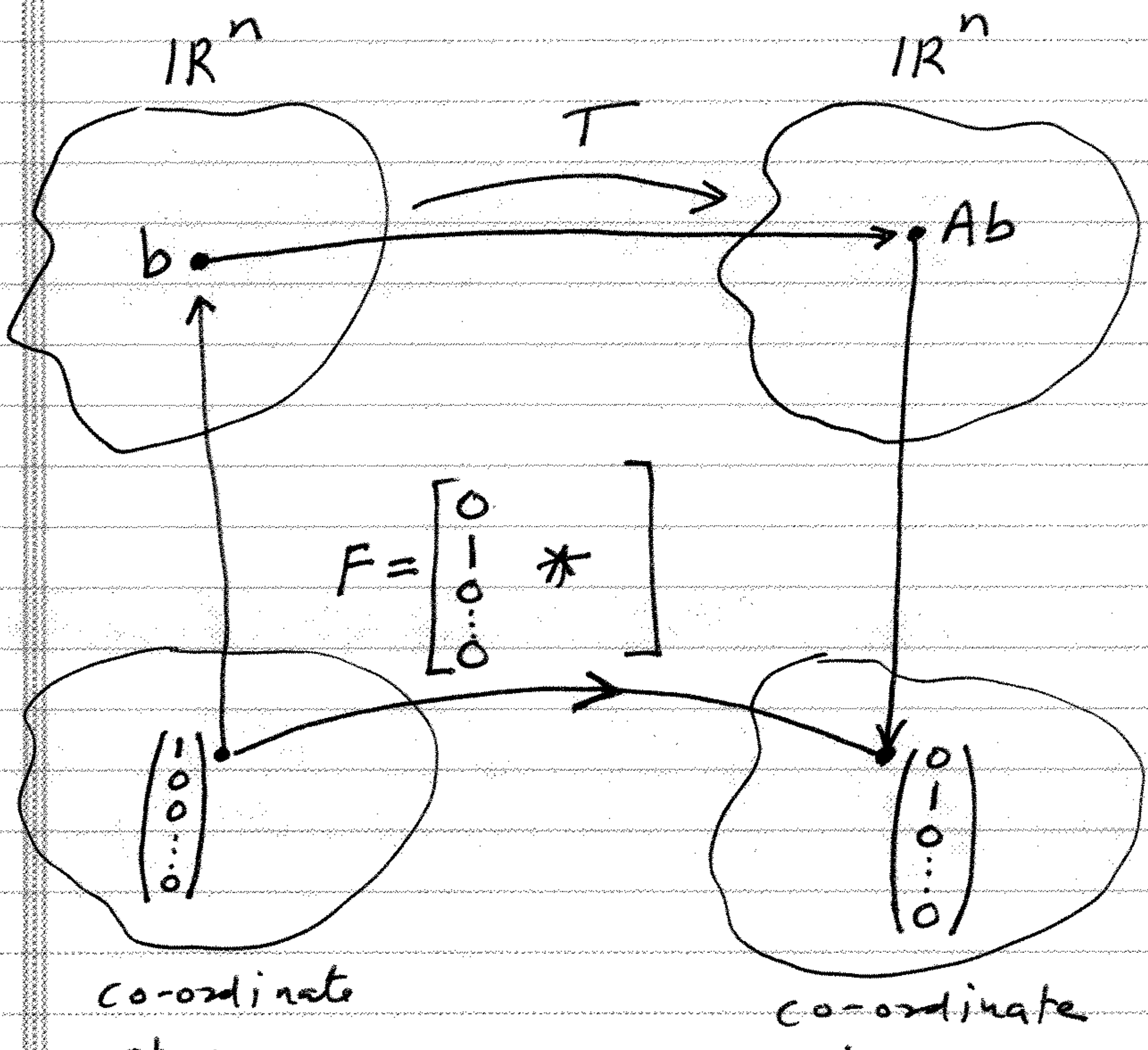
where A is any $n \times n$ matrix. We would like to represent the transformation T with respect to a special choice of basis.

Choice of basis:

Assume that $\exists$ a vector $b \in \mathbb{R}^n$ such that the set of vectors

$$B = \{b, Ab, A^2b, A^3b, \ldots, A^{n-1}b\}$$

are linearly independent. It therefore forms a basis of $\mathbb{R}^n$. We would like to represent the transformation T with respect to the basis B.



$$\text{Basis} = \{b, Ab, \ldots, A^{n-1}b\}$$

Let F be the representation of the transformation T with respect to the basis B

we know from Lec 13 that

1. i^{th} column of F is the co-ordinate vector of $T(A^{i-1}b)$ w.r.t the basis B.

The first $n-1$ columns of F are

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \times \\ 1 & 0 & \cdots & 0 & \times \\ 0 & 1 & \cdots & \vdots & \times \\ \vdots & 0 & \cdots & \vdots & \times \\ \vdots & \vdots & \cdots & 0 & \times \\ 0 & 0 & \cdots & 1 & \times \end{pmatrix}$$

This is because

$$T(A^{i-1}b) = A(A^{i-1}b) = A^{i}b$$

co-ordinate vector of $A^{i}b$ is

$$(0, 0, \ldots, 0, \underset{i+1^{th}\text{ spot}}{1}, 0, \ldots 0)^T \qquad i = 1, 2, \ldots, n-1.$$

Finally if we assume that the characteristic polynomial of A is

$$\lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_n$$

we obtain

$$A^n = -\alpha_1 A^{n-1} - \alpha_2 A^{n-2} + \cdots + (-\alpha_n) I$$

$$\Rightarrow A^n b = -\alpha_1 A^{n-1} b - \alpha_2 A^{n-2} b - \cdots - \alpha_n b.$$

$T(A^{n-1}b) = A^n b$ has co-ordinates

$$(-\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_1)^T$$

Hence the last column of F is given by

$$\begin{pmatrix} -\alpha_n \\ -\alpha_{n-1} \\ \vdots \\ \vdots \\ -\alpha_1 \end{pmatrix}$$

Thus

$$F = \begin{pmatrix} 0 & 0 & 0 & \cdot & \cdot & 0 & -\alpha_n \\ 1 & 0 & 0 & \cdot & \cdot & 0 & -\alpha_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot & 0 & -\alpha_{n-2} \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \vdots \\ \vdots & \vdots & \vdots & \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \cdot & \cdot & 0 & -\alpha_3 \\ \vdots & \vdots & \vdots & \cdot & \cdot & 0 & -\alpha_2 \\ 0 & 0 & 0 & \cdot & \cdot & 1 & -\alpha_1 \end{pmatrix}$$

is the matrix representation of the linear transformation T wrt the basis B.

Remark: If we define

$$P = \begin{pmatrix} b & Ab & A^2 b & \cdots & A^{n-1} b \end{pmatrix}$$

as an $n \times n$ matrix, we have

$$\boxed{PF = AP}$$ ← Can be verified by multiplying.

Hence $F = P^{-1} A P$

Thus we have the following theorem:

If A, b are such that the set of vectors $b, Ab, \ldots, A^{n-1}b$ are linearly independent, then the matrix A is similar to a matrix F (which is transpose of a matrix in companion form).

Remark:

Consider an o.d.e in the form

$$\dot{\underline{x}} = A\underline{x} + bf(t)$$

where $\underline{x}(t) \in \mathbb{R}^n$, A is a $n \times n$ matrix

$b \in \mathbb{R}^n$

Assume $b, Ab, A^2b, \ldots, A^{n-1}$ l.i.

Define $P = (b, Ab, \ldots, A^{n-1}b)$

Consider a new state variable $Z(t)$:

$\underline{x}(t) = P\,Z(t)$.

It follows that

$$\dot{\underline{x}} = P\dot{Z}(t)$$

But

$$\dot{\underline{x}} = A\underline{x} + bf$$

$$= APZ(t) + bf(t)$$

It follows that

$$P\dot{Z} = APZ(t) + bf(t)$$

$$\Rightarrow \dot{Z} = P^{-1}APZ(t) + P^{-1}b\,f(t)$$

$P^{-1}AP = F$ (given on page 3·5)

On the other hand

$$P^{-1}b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

because

$$b = P\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ and $P = (b, Ab, A^2b, \ldots, A^{n-1}b)$

Thus, w.r.t. the variable $Z(t)$ we have

$$\underbrace{\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{pmatrix}}_{\dot{Z}(t)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_{n-1} \\ 0 & 1 & \cdots & 0 & -\alpha_{n-2} \\ 0 & 0 & & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_2 \\ 0 & 0 & \cdots & 1 & -\alpha_1 \end{pmatrix} \underbrace{\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix}}_{Z(t)} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} f(t)$$

$$\dot{z}_1 = -\alpha_n z_n + f(t)$$
$$\dot{z}_2 = z_1 - \alpha_{n-1} z_n$$
$$\dot{z}_3 = z_2 - \alpha_{n-2} z_n$$
$$\cdots$$
$$\cdots$$
$$\dot{z}_{n-1} = z_{n-2} - \alpha_2 z_n$$
$$\dot{z}_n = z_{n-1} - \alpha_1 z_n$$

Example 1 :-

In this example, we change the basis vector set to B_1 as follows:

$$B_1 = \{v_1, v_2, \ldots, v_n\}$$

where

$$v_n = b$$

$$v_{n-1} = Ab + \alpha_1 b$$

Note: This basis vector is used to get a matrix in its companion form

$$v_{n-2} = A^2 b + \alpha_1 Ab + \alpha_2 b$$

$$v_{n-3} = A^3 b + \alpha_1 A^2 b + \alpha_2 Ab + \alpha_3 b$$

$$\cdots$$

$$\cdots$$

$$v_1 = A^{n-1} b + \alpha_1 A^{n-2} b + \alpha_2 A^{n-3} b + \cdots + \alpha_{n-1} b$$

We now represent the transformation T with respect to the new basis B_1. Let F_1 be the associated matrix representation.

The 1st column of F_1 are the co-ordinates to $T(v_1)$ w.r.t. B_1.

$$T(v_1) = A^n b + \alpha_1 A^{n-1} b + \alpha_2 A^{n-2} b + \cdots + \alpha_{n-1} A b$$

$$= \underbrace{\left(A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_{n-1} A\right)}_{= -\alpha_n I} b$$

$$= -\alpha_n b = -\alpha_n v_n$$

1st column is $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\alpha_n \end{pmatrix}$

All other columns are computed likewise

$$F_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & & -\alpha_1 \end{pmatrix}$$

Note: The companion form

If we define the matrix P_1 analogously as

$$P_1 = \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{pmatrix}$$

it follows that

$$P_1^{-1} A P_1 = F_1$$

Finally we have

$$P_1 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = v_n = b$$

$$\Rightarrow P_1^{-1} b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Remark!

As before if we consider an o.d.e in the form

$$\dot{\underline{x}} = A\underline{x} + b f(t).$$

where $\underline{x}(t) \in \mathbb{R}^n$, A is a $n \times n$ matrix $b \in \mathbb{R}^m$. Let P_1 be as in page 3·11 if we have

$$\underline{x}(t) = P_1 Z_1(t)$$

then we can write

$$\begin{aligned}
\dot{Z}_1(t) &= P_1^{-1} A P_1 Z_1(t) + P_1^{-1} b f(t) \\
&= F_1 Z_1(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} f(t) \\
&= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -\alpha_n & & & & -\alpha_1 \end{pmatrix} Z_1(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} f(t).
\end{aligned}$$

If

$$Z_1 = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}(t)$$

we have

$$\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \end{aligned}\right\}$$

chain of integrators.

$$\dot{z}_n = -\alpha_n z_1 - \alpha_{n-1} z_2 - \cdots - \alpha_1 z_n + f(t)$$

Example 2

Let A be a 4×4 matrix and let b_1, b_2 be two 4×1 vectors such that "b_1, Ab_1, b_2, Ab_2 are linearly independent"

Let us assume that

$$A^2 b_1 = \alpha_1 b_1 + \alpha_2 A b_1$$

$$A^2 b_2 = \alpha_3 b_1 + \alpha_4 Ab_1 + \alpha_5 b_2 + \alpha_6 Ab_2$$

Let T be a linear transformation

$$T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$x \longmapsto Ax \ , x \in \mathbb{R}^4.$$

We would like to represent T w.r.t. the basis

$$\{b_1, Ab_1, b_2, Ab_2\}$$

To do this we write

$$v_1 = b_1, \ v_2 = Ab_1, \ v_3 = b_2, \ v_4 = Ab_2$$

and compute

$T(v_1) = Ab_1 = v_2$

$$T(v_2) = A^2 b_1 = \alpha_1 b_1 + \alpha_2 A b_1$$
$$= \alpha_1 v_1 + \alpha_2 v_2 .$$

$T(v_3) = Ab_2 = v_4$

$T(v_4) = A^2 b_2 = \alpha_3 v_1 + \alpha_4 v_2 + \alpha_5 v_3 + \alpha_6 v_4 .$

Matrix F_1 that represents T w.r.t. basis

$$B_1 = \{ b_1, Ab_1, b_2, Ab_2 \}$$

is

$$\begin{pmatrix} 0 & \alpha_1 & 0 & \alpha_3 \\ 1 & \alpha_2 & 0 & \alpha_4 \\ 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & 1 & \alpha_6 \end{pmatrix} = F_1$$

If we define

$$P_1 = \begin{pmatrix} b_1 & Ab_1 & b_2 & Ab_2 \end{pmatrix},$$

it follows that

$$P_1^{-1} A P_1 = F_1$$

As in page 3·6, if we consider an o.d.e. in the form

$$\dot{\underline{X}} = A\underline{X} + b_1 f_1(t) + b_2 f_2(t),$$

$\underline{X}(t) \in \mathbb{R}^n$, A is an $n \times n$ matrix

$b_1, b_2 \in \mathbb{R}^n$;

We consider a new state variable

$$\underline{X}(t) = P_1 Z(t)$$

where

$$P_1 = \begin{pmatrix} b_1 & Ab_1 & b_2 & Ab_2 \end{pmatrix}.$$

It follows that

$$\dot{Z} = P_1^{-1} A P_1 Z(t) + P_1^{-1} b_1 f_1(t) + P_1^{-1} b_2 f_2(t)$$

$$= \begin{pmatrix} 0 & \alpha_1 & 0 & \alpha_3 \\ 1 & \alpha_2 & 0 & \alpha_4 \\ 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & 1 & \alpha_6 \end{pmatrix} Z(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} f_1(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} f_2(t)$$

Note that $P_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = b_1$; $P_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = b_2$.

3.17

Writing

$$Z = (z_1 \ z_2 \ z_3 \ z_4)^T$$

we have

$$\dot{z}_1 = \alpha_1 z_2 + \alpha_3 z_4 + f_1(t)$$

$$\dot{z}_2 = z_1 + \alpha_2 z_2 + \alpha_4 z_4$$

$$\dot{z}_3 = \alpha_5 z_4 + f_2(t)$$

$$\dot{z}_4 = z_3 + \alpha_6 z_4$$

Example 3:

Let us reconsider example 2 by choosing another basis given by

$$\{Ab_1 - \alpha_2 b_1, b_1, Ab_2 - \alpha_6 b_2 - \alpha_4 b_1, b_2\} = B_2$$

Defining

$v_1 = Ab_1 - \alpha_2 b_1$, $v_2 = b_1$, $v_3 = Ab_2 - \alpha_6 b_2 - \alpha_4 b_1$

$v_4 = b_2$,

we obtain

$$T(v_1) = Av_1 = A^2 b_1 - \alpha_2 Ab_1 = \alpha_1 b_1 = \alpha_1 v_2$$

$$T(v_2) = Ab_1 = v_1 + \alpha_2 b_1 = v_1 + \alpha_2 v_2$$

$$T(v_3) = Av_3 = A^2 b_2 - \alpha_6 Ab_2 - \alpha_4 Ab_1$$
$$= \alpha_3 b_1 + \alpha_5 b_2$$
$$= \alpha_3 v_2 + \alpha_5 v_4$$

$$T(v_4) = Ab_2 = v_3 + \alpha_6 b_2 + \alpha_4 b_1$$
$$= v_3 + \alpha_4 v_2 + \alpha_6 v_4$$

Matrix F_2 that represents T w.r.t.

basis

$$\{Ab_1 - \alpha_2 b_1,\ b_1,\ Ab_2 - \alpha_6 b_2 - \alpha_4 b_1,\ b_2\}$$

is

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha_5 & \alpha_6 \end{array}\right) = F_2$$

Note: Each 2x2 diagonal block matrix in F_2 is in the companion form.

If we define

$$P_2 = \left(Ab_1 - \alpha_2 b_1,\ b_1,\ Ab_2 - \alpha_6 b_2 - \alpha_4 b_1,\ b_2\right)$$

we have

$$P_2^{-1} A P_2 = \left(\quad F_2 \quad\right)$$

$$P_2^{-1} b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \quad P_2^{-1} b_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The ode on page 3.16 would reduce to

$$\dot{Z} = F_2 Z + \begin{pmatrix} 0 \\ f_1(t) \\ 0 \\ f_2(t) \end{pmatrix}$$

where

$$x(t) = P_2 Z(t).$$

Hence

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \alpha_4 z_4 + f_1(t).$$

$$\dot{z}_3 = z_4$$

$$\dot{z}_4 = \alpha_5 z_3 + \alpha_6 z_4 + f_2(t)$$

Example 4:

Let

$$A = \begin{pmatrix} 0 & -1 & -1 & 1 \\ -5 & 6 & 7 & -6 \\ -7 & 8 & 9 & -7 \\ -7 & 8 & 10 & -7 \end{pmatrix}$$

$$B = (b_1 \quad b_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

If we construct the set of basis vectors described in Example 3, we have

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}; \; v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}; \; v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}; \; v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The vectors v_1, v_2, v_3, v_4 are obtained by computing $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ as follows

$\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3,\ \alpha_4 = 4,\ \alpha_5 = 5,\ \alpha_6 = 6.$

We define P_2 as

$$P_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and obtain

$$P_2^{-1} A P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 6 \end{pmatrix}$$

which has the block companion structure.

3.23

$$P_2^{-1}B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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